Name: Ibrahim Intabli

Student ID: 201003203

Instructions:

1. This exam has 6 pages. Please make sure you have all pages.
2. The point value of each problem occurs to the left of the problem.
3. **You must show correct work to receive credit.** Correct answers with inconsistent work or with no justification will not be given credit.
4. Books, notes and calculators are not allowed.
5. **Turn off and put away all cell phones.**

Page	Points	Points Possible
2	4	5
3	9	9
4	6	6
5	4	5
6	3	5
Total	26	30

Name: Ibrahim Intabli.

1. (3 pts) List, up to isomorphism, all abelian groups of order 600. Which of these groups is cyclic?

First, we find the prime factorization of 600:

600	2
300	2
150	2
75	3
25	5
5	5
1	

$\Rightarrow 600 = 2^3 \times 3 \times 5^2$.

So, any abelian group of order 600 must be isomorphic to one of the following:

- $\mathbb{Z}_8 \times \mathbb{Z}_3 \times \mathbb{Z}_{25}$
- $\mathbb{Z}_8 \times \mathbb{Z}_3 \times \mathbb{Z}_5 \times \mathbb{Z}_5$
- $\mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_3 \times \mathbb{Z}_{25}$
- $\mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_3 \times \mathbb{Z}_5 \times \mathbb{Z}_5$
- $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_{25}$
- $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_5 \times \mathbb{Z}_5$.

2

2. (2 pts) Find the order of $(\mathbb{Z}_{12} \times \mathbb{Z}_{16})/ < (10, 12) >$.

index $(\mathbb{Z}_{12}\times\mathbb{Z}_{16}) : \langle(10,12)\rangle = \left|(\mathbb{Z}_{12}\times\mathbb{Z}_{16})/\langle(10,12)\rangle\right| = \dfrac{|\mathbb{Z}_{12}\times\mathbb{Z}_{16}|}{|\langle(10,12)\rangle|}$

2

$$= \frac{12 \times 16}{\mathrm{lcm}\left(\frac{12}{\gcd(12,10)}, \frac{16}{\gcd(16,12)}\right)}$$

$$= \frac{12 \times 16}{\mathrm{lcm}\left(\frac{12}{2}, \frac{16}{4}\right)}$$

$$= \frac{12 \times 16}{\mathrm{lcm}(6,4)} = \frac{12 \times 16}{12} = 16.$$

Math 241 Quiz 2 Fall 2010

Name: Ibrahim Intabli

3. (3 pts) Find the order of $(4,2)+<(4,4)>$ in $(\mathbb{Z}_6 \times \mathbb{Z}_8)/<(4,4)>$.

~~8 (4,2) = (24 mod 6, 12 mod 8~~

$\langle (4,4) \rangle = \{(4,4),(2,0),(0,4),(4,0),(2,4),(0,0)\}$

note that $2.(4,2) = (2,4) \in \langle (4,4) \rangle$

so $2 \times [(4,2) + \langle (4,4) \rangle] = \langle (4,4) \rangle$,

hence, the order we are looking for is 2.

3

4. (3 pts) Show that $\mathbb{Z}_3 \times \mathbb{Z}_6$ is not isomorphic to $\mathbb{Z}_{18}$.

~~$\mathbb{Z}_3 \times \mathbb{Z}_6$~~

as 3 and 6 fail to be relatively prime n's, since:

$\mathbb{Z}_{18} \cong \mathbb{Z}_9 \times \mathbb{Z}_2$

$\mathbb{Z}_3 \times \mathbb{Z}_6 \cong \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_2$

but $\mathbb{Z}_3 \times \mathbb{Z}_3 \not\cong \mathbb{Z}_9$ as 3 & 3 aren't relatively prime.

3

5. (3 pts) Find a nontrivial homomorphism $\phi : \mathbb{Z}_{24} \to \mathbb{Z}_{16}$.

consider the homomorphism ϕ:

$\phi(0) = 0$

$\phi(1) = 4$

$\phi(2) = 8$

$\phi(3) = 12$

$\phi(4) = 0$

$\vdots$

$\phi(n) = (n \bmod 4) \times 4$ where $n \in \mathbb{Z}_{24}$

ϕ is a hom.

$\phi : \mathbb{Z}_{24} \to \langle 4 \rangle$

ϕ is onto $\langle 4 \rangle$ of $\mathbb{Z}_{16}$

so ϕ is a hom. from $\mathbb{Z}_{24}$ into $\mathbb{Z}_{16}$

note that $|\langle 4 \rangle| = 4$ divides both $|\mathbb{Z}_{24}| = 24$ & $|\mathbb{Z}_{16}| = 16$

3

Name: Ibrahim Intabli.

6. Let S_3 be the group of permutations on 3 letters.

(a) (3 pts) Find the left cosets of $<(1,2)>$ in S_3.

$$\langle (1,2) \rangle = \{ (1,2) , \rho_0 \}$$

id. permutation

~~so~~ since $|\langle(1,2)\rangle| = 2$, $\left|S_3 / \langle(1,2)\rangle\right| = \frac{|S_3|}{2} = \frac{6}{2} = 3$

so there are 3 cosets of $\langle(1,2)\rangle$ in S_3:

3

- $(1,2)\langle(1,2)\rangle = \rho_0 \langle(1,2)\rangle = \langle(1,2)\rangle = \{(1,2), \rho_0\}$
- $(1,3)\langle(1,2)\rangle = \{(1,3)(1,2), (1,3)\rho_0\} = \{(1\ 2\ 3), (1,3)\}$
- $(2,3)\langle(1,2)\rangle = \{(2,3)(1,2), (2,3)\rho_0\} = \{(1\ 3\ 2), (2,3)\}$

(b) (3 pts) Is $<(1,2)>$ a normal subgroup of S_3?

No, because $\langle(1,2)\rangle.(1,3) = \{(1,2)(1,3), \rho_0(1,3)\}$

$= \{(1\ 3\ 2), (1,3)\}$

$\neq (1,3)\langle(1,2)\rangle$

3

so, there is an element (at least one is enoagh to prove false) in S_3 for which

$a . \langle(1,2)\rangle \neq \langle(1,2)\rangle . a$

~~righ~~ left coset $\neq$ right coset

Name: Ibrahim Intabli

7. (5 pts) Let H be a subgroup of a group G and let $N(H) = \{g \in G | gHg^{-1} = H\}$. Show that $N(H)$ is a subgroup of G.

$N(H) = \{g \in G \mid gHg^{-1} = H\}$

must prove $N(H) \leqslant G$

i) closure:

let $g_1, g_2 \in N(H)$

then $g_1 H g_1^{-1} = g_2 H g_2^{-1} = H$

4

must prove $g_2 g_1 \in N(H)$; i.e.,

$$g_2 g_1 H (g_2 g_1)^{-1} = H$$

since $g_1 H g_1^{-1} = \cancel{g_2^{-1} H g_2}$; then for $h \in H$,

we have $g_1 h g_1^{-1} = g_2^{-1} h' g_2$ for some $h' \in H$

$$g_2 g_1 h g_1^{-1} g_2^{-1} = h' \in H$$

$$g_2 g_1 h (g_2 g_1)^{-1} = h' \in H$$

ii) identity element:

e_G, the id. elt. of G, $\in N(H)$

since $e_G h (e_G)^{-1} = h \in H$ $\forall h \in H$ and $h = ehe^{-1} \in gHg^{-1}$

therefore $e_G H (e_G)^{-1} = H$.

iii) inverses: let $g \in N(H) \Rightarrow g^{-1}$ exists in group G

$\Rightarrow gHg^{-1} = H$

$\Rightarrow ghg^{-1} = h'$ for some $h, h' \in H$

NOT scratch:

$\Rightarrow \; g^{-1} h' g = h \in H$ for $h', h \in H$

$\Rightarrow \; g^{-1} H g = H \Rightarrow g^{-1} \in N(H)$

(remember the 3 equivalent statements for normal subgroups)

Hence $N(H) \leq G$. $\blacksquare$

Name: Ibrahim Intabli

3. (5 pts) Let H be a subgroup of a group G. Show that gH is a subgroup of G if and only if $g \in H$.

must prove: $gH \leqslant G \iff g \in H$

($\Leftarrow$) suppose $g \in H$

then $gH = \{gh \mid h \in H\}$

i) closure: ~~let~~ let $k_1, k_2 \in gH$

then $k_1 = gh_1$ & $k_2 = gh_2$ for some $h_1, h_2 \in H$

$k_1 k_2 = g\underbrace{h_1 g}_{\in H} h_2$

3

ii) identity elt.: $e_G \in gH$ why?

iii) inverses: let $k \in gH$

then $k = gh$ for some $h \in H$.

$k^{-1} \in G$ as G is a group.

$k^{-1} = (gh)^{-1} = h^{-1}g^{-1} \in H$

$\Rightarrow h^{-1}g^{-1} = h' \in H$